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LETTER TO THE EDITOR

Series expansion study of the distribution of currents in the elements of a random diode–insulator network

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Abstract. The critical exponent ζ_k of the kth moment of the current distribution for random diode-insulator networks on the square and simple cubic lattices is calculated for a range of k using low density series expansion techniques. It is also shown that $\zeta_1 = \nu_\parallel$ where $\nu_\parallel$ is the critical exponent of the parallel connectedness length for directed percolation. Our values of ζ_k for the simple cubic lattice are well fitted by a simple exponential formula with $(\zeta_k - 1)/(\zeta_{k+1} - 1) = \frac{1}{2}$. The Skal-Shklovskii scaling relation for the conductivity is generalised to the kth moment and it follows that the exponent κ describing the divergence of flicker noise is given by

$$\kappa \doteq (d-1)\nu_\perp + \nu_\parallel + \zeta_4 - 2\zeta_2 .$$

This leads to the estimates

$$\kappa = \begin{cases} 1.17 \pm 0.03 & \text{square lattice} \\ 1.46 \pm 0.06 & \text{simple cubic lattice.} \end{cases}$$

The current distribution in a random conductor-insulator network has recently been discussed in depth by Rammal *et al* [1, 2] where further references to the relevant background may be found. Here we investigate diode-insulator networks in which each edge of a graph G is a diode with probability p and an insulator with probability $1-p$ independently of all other edges. In the configuration in which the diodes correspond to the edges of the subgraph G' we let $i_e(G')$ be the current through edge e when a current I is passed between vertices u and v except when there is no conducting path from u to v in which case $i_e(G') = 0$ for all e. For $k \geqslant 0$ we define, following de Arcangelis *et al* [3], the kth moment of the current distribution by

$$L_k(u, v; G') = \sum_{e \in G'} [i_e(G')/I]^k \qquad (1)$$

and denote its value when averaged over all configurations (percolation average) by $L_k(u, v; p)$.

In the case when G is a lattice graph we suppose that all diodes, when present, are directed so as to have a positive component parallel to some fixed preferred direction [4]. For such a graph there is a probability (the critical probability for directed

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percolation) below which the expected number of sites reachable from a given site by a conducting path is finite and above which it is infinite. The mean cluster size $S(p)$ is the expected number of reachable sites given that this is finite. In the percolation average which determines $L_k(u, v; p)$ we give zero weight to configurations in which the number of reachable sites is infinite. For a lattice graph in which all sites are equivalent the sum of $L_k(u, v; p)$ over v is independent of u and will be denoted by $\chi_k(p)$.

For any G' in which u and v are connected the union of all paths from u to v on G' is known as the u-v backbone for G' and denoted by $b(G')$. The value of $L_k(u, v; G')$ clearly depends only on $b(G')$ since $i_e(G') = 0$ for any edge not in $b(G')$. It has been noted [2] that for undirected percolation $L_0(u, v; G')$ is the number of edges in $b(G')$, $L_2(u, v; G')$ is the resistance measured between vertices u and v of $b(G')$ and $L_\infty(u, v; G')$ is the number of nodal edges in $b(G')$. Also L_4/L_2^2 is a measure of the flicker noise amplitude [2]. These results are valid for directed as well as undirected percolation. We now show that, for the diode-insulator problem on the hypercubic lattice, $L_1(u, v; G')$ is the length t_{uv} of the shortest path or chemical distance from u to v whenever u and v are connected. Since for this problem all conducting paths from u to v have the same length, t_{uv} is this common length and the result is valid for any directed lattice with this property. Also t_{uv} is proportional to the distance from u to v measured parallel to the preferred direction.

The proof is as follows. For any subgraph G' the edges of $b(G')$ may be partitioned into subsets $E_1, E_2, \ldots, E_t$ where the edges E_s have a final vertex which is s steps away from u. For given E_s any charge passing from u to v must pass through exactly one of the arcs in E_s so that the sum of $i_e(G')$ over the arcs of E_s must be equal to I; hence

$$L_1(u, v; G') = \sum_{e \in b(G')} (i_e(G')/I) = \sum_{s=1}^{t_{uv}} (1/I) \sum_{e \in E_s} i_e(G') = t_{uv}. \tag{2}$$

A similar result has been obtained by Blumenfeld and Aharony [5] for non-linear random resistor networks. A correspondence between the resistance of a non-linear network and the moments of the current distribution for a linear network has been demonstrated for hierarchical lattices by de Arcangelis *et al* [6]. If this relation were exact for real lattices then our result would be equivalent to that of Blumenfeld and Aharony. However, it is easily shown that (2) is not exact when paths between u and v of different length occur and it is therefore not exact for undirected percolation.

The average value of $L_k(u, v; p)$ over all pairs of lattice sites is given by $\mathscr{L}_k(p) = \chi_k(p)/S(p)$. The above interpretations imply that $\mathscr{L}_2(p)$ is the point-to-point resistance function considered in reference [7] and $\mathscr{L}_1(p)$ is a measure of $\xi_\parallel(p)$, the connectedness length [8, 9] parallel to the preferred direction. Coniglio [10] has shown that $\mathscr{L}_\infty(p)$, the expected number of nodal edges, diverges at p_c with critical exponent $\zeta_\infty = 1$ and since $\mathscr{L}_k(p) \geq \mathscr{L}_\infty(p)$,

$$\mathscr{L}_k(p) \sim (p_c - p)^{-\zeta_k} \tag{3}$$

where for $k' > k$, $1 < \zeta_{k'} \leq \zeta_k$. It has been shown [9] that for directed percolation the expected number of backbone edges $\mathscr{L}_0(p)$ is equal to $S(p)$ so that $\zeta_0 = \gamma$ and from our result for $\mathscr{L}_1(p)$, $\zeta_1 = \nu_\parallel$. These exponents and the resistive exponent $\zeta_2 = \zeta_R$ have been previously calculated for directed percolation on the square and simple cubic lattices by low density series expansion methods [9, 11, 12]. In this letter we extend these calculations to higher values of k.

The moments $L_k(u, v; G')$ have the following two important properties in common with the resistance $R_{uv}(G')$ between u and v.

(i) L_k depends only on the backbone of G' since no current passes through the arcs not in the backbone.

(ii) If G' has a path from u to v and is the series combination of graphs G'_1 and G'_2 which have vertex w in common then $L_k(u, v; G') = L_k(u, w; G'_1) + L_k(w, v; G'_2)$.

Functions with properties (i) and (ii) are called additive backbone functions and it has been shown [11] that the susceptibilities corresponding to such functions for percolation on directed lattices may be factorised, $\chi_k(p) = \Psi_k(p)S(p)^2$, where $\Psi_k(p)$ is the contribution to the cluster expansion from non-nodal backbones. It has also been shown [13] that

$$\Psi_k(p) = \sum_i L_k(b_i)\psi_i(p) \tag{4}$$

where the sum is over all possible backbones b_i with initial root at u. $\psi_i(p)$ is a polynomial of order p^α, where α is the number of arcs in the smallest non-nodal backbone which contains b_i and is known as a generalised perimeter polynomial. It has the useful property of being independent of k and since $\Psi_1(p)$ may be found by an independent transfer matrix method [9] we have a good check on the calculation of the ψ_i.

We have obtained the series expansions of $\chi_k(p)$ on the square and simple cubic lattices for various values of k to order p^{17} and p^{11} respectively. The number of non-isomorphic non-nodal backbone graphs required was 320 for the square lattice and 28 for the simple cubic lattice and the number of non-isomorphic backbones which are subgraphs of these graphs is 1177 and 46 respectively. A list of these graphs for the square lattice as far as 12 edges may be found in reference [13]. The series coefficients for $\chi_4(p)$ are given in table 1 and we take this opportunity of correcting minor errors arising from real to rational conversion in the last coefficients of our

Table 1. Coefficients in the low density expansion of $\chi_4(p) = \Sigma_{n=1}^{\infty} a_n p^n$.

	Square lattice	Simple cubic lattice
n	a_n	a_n
1	0.200 000 000 000 000 000 000 000 00D+01	0.300 000 000 000 000 000 000 000 00D+01
2	0.800 000 000 000 000 000 000 000 00D+01	0.180 000 000 000 000 000 000 000 00D+02
3	0.240 000 000 000 000 000 000 000 00D+02	0.810 000 000 000 000 000 000 000 00D+02
4	0.602 500 000 000 000 000 000 000 00D+02	0.312 750 000 000 000 000 000 000 00D+03
5	0.141 000 000 000 000 000 000 000 00D+03	0.112 950 000 000 000 000 000 000 00D+04
6	0.303 750 000 000 000 000 000 000 00D+03	0.382 387 500 000 000 000 000 000 00D+04
7	0.639 976 400 000 000 000 000 000 00D+03	0.126 036 084 000 000 000 000 000 00D+05
8	0.127 490 560 000 000 000 000 000 00D+04	0.400 021 410 000 000 000 000 000 00D+05
9	0.252 768 808 369 324 008 990 819 78D+04	0.125 315 555 077 620 823 570 845 19D+06
10	0.480 288 523 884 002 847 463 921 25D+04	0.382 774 804 741 288 056 069 767 01D+06
11	0.914 566 333 965 692 816 279 676 95D+04	0.116 130 497 504 609 022 984 862 51D+07
12	0.168 173 114 084 769 819 902 381 79D+05	
13	0.311 755 821 203 955 838 403 303 78D+05	
14	0.558 363 782 428 447 170 318 662 77D+05	
15	0.101 656 399 753 088 407 250 517 17D+06	
16	0.178 327 794 467 743 366 587 878 51D+06	
17	0.320 003 545 056 116 177 684 968 32D+06	

Table 2. Padé approximant estimates of γ_k, γ, $\nu_\parallel$ and $\nu_\perp$.

	γ_k		
k	Square lattice ($p_c = 0.644\,701 \pm 0.000\,012$†)	Simple cubic lattice ($p_c = 0.3814 \pm 0.0007$‡)	Exact relation
0.0	$4.559 + 111\Delta p_c \pm 0.015$	$3.067 + 59\Delta p_c \pm 0.041$	$\gamma_{BB} = 2\gamma$
0.5	$4.247 + 93\Delta p_c \pm 0.015$	$2.902 + 59\Delta p_c \pm 0.024$	
1.0	$3.997 + 71\Delta p_c \pm 0.018$	$2.789 + 57\Delta p_c \pm 0.020$	$\nu_\parallel + \gamma$
1.5	$3.805 + 68\Delta p_c \pm 0.017$	$2.712 + 42\Delta p_c \pm 0.018$	
2.0	$3.660 + 65\Delta p_c \pm 0.008$	$2.659 + 42\Delta p_c \pm 0.012$	γ_R
3.0	$3.479 + 54\Delta p_c \pm 0.013$	$2.597 + 37\Delta p_c \pm 0.011$	
4.0	$3.387 + 48\Delta p_c \pm 0.016$	$2.567 + 35\Delta p_c \pm 0.009$	
5.0	$3.337 + 48\Delta p_c \pm 0.009$	$2.552 + 34\Delta p_c \pm 0.010$	
6.0	$3.310 + 52\Delta p_c \pm 0.009$	$2.544 + 33\Delta p_c \pm 0.012$	
8.0	$3.289 + 48\Delta p_c \pm 0.009$	$2.538 + 33\Delta p_c \pm 0.008$	
10.0	$3.282 + 56\Delta p_c \pm 0.008$		
12.0	$3.279 + 53\Delta p_c \pm 0.005$		
γ	$2.277\,21 + 90\Delta p_c \pm 0.000\,01$†	$1.533 + 34\Delta p_c \pm 0.002$‡	
$\nu_\parallel$	$1.7332(5) + 68\Delta p_c \pm 0.0001$†	$1.264 + 16\Delta p_c \pm 0.002$‡	
$\nu_\perp$	$1.097 + 64\Delta p_c \pm 0.001$†	$0.706 + 14\Delta p_c \pm 0.002$‡	
κ	$1.174 + 140\Delta p_c \pm 0.03$	$1.458 + 29\Delta p_c \pm 0.041$	

† Reference [13].
‡ From reanalysis of the series of reference [7] using correction to scaling analysis.

previously published series [11, 12] for $\chi_R(p) = \chi_2(p)$. The numerators in the last coefficients of $\Psi_R(p)$ should be: on the square lattice $a_{17}(\text{Num}) = 11\ 188\ 788\ 611\ 053\ 562\ 394\ 217\ 805\ 041\ 069\ 668\ 230\ 454\ 688\ 054\ 618\ 909\ 687\ 392\ 954\ 944\ 236\ 841\ 993\ 282\ 790\ 166\ 252\ 801$ and on the simple cubic lattice $a_{11}(\text{Num}) = 52\,662\,427\,739\,944$. The denominators are correct. The corresponding corrected $\chi_R(p)$ coefficients are $b_{17} = 373\,220.825\,206\,888\,192\,393\,833\,02$ and $b_{11} = 1220\,941.342\,557\,058\,361\,295\,4618$. No significant change in our estimates of γ_R was caused by these errors and no such errors occurred in our calculations for the undirected square lattice [14].

We have determined the critical exponent γ_k of $\chi_k(p)$ for a range of values of k using the series analysis method of Adler *et al* [15] and the results are shown in table 2. The quoted errors allow for a correction to scaling exponent Δ_1 in the range $|\Delta_1 - 1| \leq 0.03$ for the square lattice [16] and $|\Delta_1 - 1.09| \leq 0.11$ for the simple cubic. The coefficient of Δp_c gives the sensitivity to change in the p_c estimate. In figure 1 we plot $\zeta_k = \gamma_k - \gamma$ using the values of γ_k in table 2 and fit the data first to an exponential curve through the points $k = 0$ and $k = 1$ and second to a curve of the type discussed in reference [3] through the point $k = 0$. Using the above results for ζ_0, ζ_1 and ζ_∞ together with an exponential assumption gives $\zeta_k = 1 + (\gamma - 1)\alpha^k$ where $\alpha = (\nu_\parallel - 1)/(\gamma - 1)$ and the curves are derived using the values of $\nu_\parallel$ and γ in table 2. For the simple cubic lattice the exponential curve is an excellent fit and all the data are consistent with a ratio $\alpha = \frac{1}{2}$. The error bars in the figure exclude the contribution from the uncertainty of p_c but a change of p_c within our quoted range does not change the above conclusion. The same ratio occurs asymptotically for $k \to \infty$ in the formula of reference [3]. The value of λ in the latter formula which gives our estimated ζ_0 is $\lambda = 0.722$. For the square lattice neither curve is a good fit but the curve of reference [3] has the merit of only being adjusted at one point. The value $\alpha = 0.574$ which we

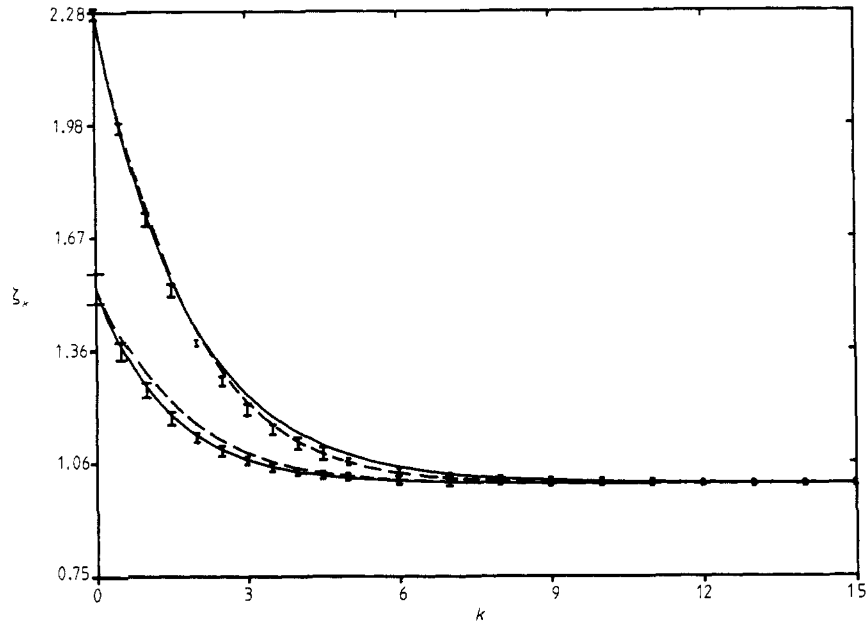

Figure 1. Plot of ζ_k against k for the square lattice (upper points) and simple cubic lattice (lower points). The full curves represent an exponential fit at the points $k=0$ and $k=1$ and the broken curves are of the type of reference [3] fitted at $k=0$.

have used was obtained from the much longer series of reference [16] whereas a better fit could be obtained with $\alpha = 0.53$. The value of λ for the square lattice is 1.114.

In order to relate ζ_k to the critical exponents for a 'parallel plate' geometry we consider a sample of the lattice in the form of a hypercube of side $L \gg \xi_{\parallel}(p)$ with preferred direction parallel to one of the cube axes. The vertices in the two $(d-1)$-dimensional hyperfaces of the cube perpendicular to the preferred direction are maintained at equal potential by 'plates' of high conductivity. This is equivalent to identifying all the vertices in each hyperface, the resulting terminal vertices being called u and v. We denote by $\Lambda_k(p, L)$ the value of $L_k(u, v; p)$ for this geometry. For $p < p_c$ the probability of a conducting path between the faces tends to zero as $L \to \infty$ and hence $\Lambda_k \to 0$. For $p > p_c$ we generalise the argument of Skal and Shklovskii [17] for the conductivity of a random conductor-insulator network. Divide the sample into rectangular supercells of length $\xi_{\parallel}(p)$ in the preferred direction and width equal to the perpendicular connectedness length $\xi_{\perp}(p)$. Writing the sum over arcs in the definition of $\Lambda_k(p, L)$ as a sum over arcs within a given supercell followed by a sum over supercells gives

$$\Lambda_k(p) = \sum_{\text{supercells}} E\left(\sum_{e \in \text{supercell}} [i_e(G')/I]^k\right) \tag{5}$$

where E is the expected value or percolation average.

Neglecting the variation of the expected value between supercells gives

$$\Lambda_k(p, L) \sim [L/\xi_{\perp}(p)]^{d-1}[L/\xi_{\parallel}(p)][i/I]^k E\left(\sum_{e \in \text{supercell}} [i_e(G')/i]^k\right) \tag{6}$$

where i is the current through a given supercell (neglecting fluctuations) and the first two prefactors together give the number of supercells. Now $i/I=(\xi_\perp(p)/L)^{d-1}$ and estimating the last term by $\mathscr{L}'_k(p)$ the average value of $L_k(u,v;p)$ over pairs of sites at distances of order $\xi_\parallel(p)$ gives

$$\Lambda_k(p,L)\sim[\xi_\perp(p)/L]^{(d-1)(k-1)}[L/\xi_\parallel(p)]\mathscr{L}'_k(p). \qquad (7)$$

Assuming that $\mathscr{L}'_k(p)\sim\mathscr{L}_k(p)$ and that $\mathscr{L}_k(p)$, $\xi_\parallel(p)$ and $\xi_\perp(p)$ have the same critical exponents above and below p_c we find, for $p>p_c$, that

$$\Lambda_k(p,L)\sim(p-p_c)^{t_k} \qquad (8)$$

where

$$t_k=(d-1)(k-1)\nu_\perp-\nu_\parallel+\zeta_k. \qquad (9)$$

For $k=2$ this reduces to the formula of Redner [18] for the conductivity exponent t_+ and for general k the value of t_k for a conductor-insulator system is obtained by setting $\nu_\perp=\nu_\parallel=\nu$.

Rammal *et al* [1] considered the noise generated by independent fluctuations in the diode resistances under constant external current conditions and took as a measure of this noise

$$\mathscr{S}_{uv}(G')=\frac{1}{R_{uv}^2}\langle(\delta R_{uv})^2\rangle=\frac{1}{R_{uv}^2}\sum_{e\in G'}\left(\frac{\partial R_{uv}}{\partial r_e}\right)_I^2\rho_e \qquad (10)$$

where $\langle\ldots\rangle$ is the average over the distribution of the resistance variables r_e, $\rho_e^2=\langle(\delta r_e)^2\rangle$ and

$$R_{uv}(G')=\sum_{f\in G'}r_f[i_f(G')/I]^2. \qquad (11)$$

It was shown in [1] that if $r_e=r$ and $\rho_e=\rho$ for all e then

$$\mathscr{S}_{uv}(G')=(\rho/r)^2L_4(u,v;G')/[L_2(u,v;G')]^2. \qquad (12)$$

Denoting the percolation average of this quantity for the parallel plate geometry by $\bar{\mathscr{S}}(p,L)$, scaling arguments in the neighbourhood of p_c lead to

$$\bar{\mathscr{S}}(p,L)\sim\Lambda_4(p,L)/(\Lambda_2(p,L))^2\sim(p-p_c)^{-\kappa} \qquad (13)$$

where, using (9),

$$\kappa=t_4-2t_2=(d-1)\nu_\perp+\nu_\parallel+\zeta_4-2\zeta_2. \qquad (14)$$

This result for the conductor-insulator system for which $\nu_\perp=\nu_\parallel=\nu$ is equivalent to $\kappa=(d+2x_1-x_2)\nu$ which may be obtained by combining the equations of Rammal *et al* [2] where x_n is defined by $\Lambda_{2n}(p_c,L)\sim L^{-x_n}$. The identification $x_n=-\zeta_{2n}/\nu$ may be made by finite-size scaling arguments with the result $\kappa=d\nu-\zeta_s$ where $\zeta_s=2\zeta_2-\zeta_4$.

Using the values of γ_2, γ_4, γ, $\nu_\parallel$ and $\nu_\perp$ from table 2 we find from (14) the value of κ in table 2. We believe these to be the first estimaes of this exponent for the diode-insulator problem.

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